Lecture 19: Hurewicz Theorem

Hurewicz Theorem connects homotopy groups with homology groups. Recall that

$$\tilde{\mathrm{H}}_n(S^n) = \mathbb{Z}.$$

Let us fix generators

$$i_n \in \tilde{\mathrm{H}}_n(S^n)$$

which are compatible with the isomorphisms

$$\tilde{\mathrm{H}}_n(S^n) = \mathrm{H}_n(D^n, S^{n-1}) = \tilde{\mathrm{H}}_{n-1}(S^{n-1}).$$

Definition

For $n \geq 1$, the Hurewicz map is

$$\rho_n : \pi_n(X) \to \mathrm{H}_n(X) \quad \text{by sending} \quad [f : S^n \to X] \to f_*(i_n).$$

Proposition

The Hurewicz map is a group homomorphism.

Proof: Let $[f], [g] \in \pi_n(X)$, their product in $\pi_n(X)$ is represented by

$$S^n \xrightarrow{\varphi} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \to X.$$

Here the map φ shrinks the equator S^{n-1} of S^n to a point, and

$$S^n / S^{n-1} = S^n \vee S^n.$$

Apply $\mathrm{H}_n(-)$ we get

$$\mathrm{H}_n(S^n) \xrightarrow{\varphi_*} \mathrm{H}_n(S^n) \oplus \mathrm{H}_n(S^n) \xrightarrow{f_* \oplus g_*} \mathrm{H}_n(X) \oplus \mathrm{H}_n(X) \xrightarrow{\text{sum}} \mathrm{H}_n(X).$$

Observe $\varphi_* : \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n) \oplus \mathrm{H}_n(S^n)$ is the diagonal map

$$x \to x \oplus x.$$

It follows that

$$\rho_n([f][g]) = f_*(i_n) + g_*(i_n) = \rho_n(f) + \rho_n(g).$$

□

Let G be a group. Given $g, h \in G$, we define its commutator by

$$[g, h] = ghg^{-1}h^{-1}.$$

Definition

We define the conmmutator subgroup $[G, G]$ to be the subgroup of G generated by all the commutators.

- $[g, h]^{-1} = [h, g]$
- $a[g, h]a^{-1} = [aga^{-1}, aha^{-1}]$. So $[G, G]$ is a normal subgroup.

Definition

The quotient group $G_{ab} = G/[G, G]$ is an abelian group, which is called the abelianization of G. The quotient map

$$G \to G_{ab}$$

is called the abelianization homomorphism, which is an isomorphism if G is an abelian group.

Proposition

The abelianization defines a functor

$$\mathcal{A} : \mathbf{Group} \to \mathbf{Ab}$$

which is left adjoint to the inclusion functor

$$j : \mathbf{Ab} \to \mathbf{Group}.$$

$$\mathcal{A} : \mathbf{Group} \rightleftarrows \mathbf{Ab} : j.$$

Example

If G is a free group, then G_{ab} is a free abelian group.

Example

The commutator subgroup of $GL_n(\mathbb{R})$ is $SL_n(\mathbb{R})$.

Example

The commutator subgroup of the symmetric group S_n is the alternating group A_n.

Theorem (Hurewicz Theorem)

Let X be a path-connected space which is $(n-1)$-connected $(n \geq 1)$. Then the Hurewicz map

$$\rho_n : \pi_n(X) \to \mathrm{H}_n(X)$$

is the abelianization homomorphism.

Explicitly, Hurewicz Theorem has the following two cases.

1. If $n = 1$, then $\rho_1 : \pi_1(X) \to \mathrm{H}_1(X)$ induces an isomorphism
$$\pi_1(X)_{\mathsf{ab}} \xrightarrow{\cong} \mathrm{H}_1(X).$$
2. If $n > 1$, then $\rho_n : \pi_n(X) \to \mathrm{H}_n(X)$ is an isomorphism.

Before we prove the Hurewicz Theorem, we first prepare some useful propositions.

Proposition

Let $f: X \to Y$ be a weak homotopy equivalence. Then

$$f_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

is an isomorphism for all n.

Proof

We can assume f is a cofibration. Then $\pi_n(Y, X) = 0$ for all n. Let $\sigma = \sum_i n_i \sigma_i$ represent an arbitrary element of $\mathrm{H}_n(Y, X)$ where

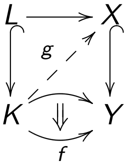
$$\sigma_i : \Delta^n \to Y, \quad \partial\sigma \in X.$$

We can use the simplexes of σ_i's to build up a finite CW complex K with a subcomplex L, and a map

$$f : K \to Y, \quad \varphi(L) \subset X$$

such that $[\sigma] = f_*[\gamma]$ is the image of an element $[\gamma] \in \mathrm{H}_n(K, L)$.

Since $X \hookrightarrow Y$ is an ∞-equivalence, f is homotopic relative L to a map g that sends K into X.



So $[\sigma] = g_*[\gamma] = 0$. It follows that

$$\mathrm{H}_n(Y, X) = 0$$

for all n. This proves the proposition. $\square$

Proposition

Let $Y = \bigvee S^n$ is a wedge of spheres ($n \geq 1$). Then

$$\pi_n(Y) \to \tilde{\mathrm{H}}_n(Y)$$

is the abelianization homomorphism.

Proof.

If $n > 1$, then $\pi_n(Y) = \tilde{\mathrm{H}}_n(Y) = \bigoplus \mathbb{Z}$.

If $n = 1$. then $\pi_1(Y)$ is a free group, $\mathrm{H}_1(Y)$ is a free abelian group which is the abelianization of $\pi_1(Y)$.

□

Proof of Hurewicz Theorem

We can assume X is a CW complex. Otherwise we replace X by a weak homotopic equivalent CW complex, which has the same homotopy and homology groups. The construction of CW approximation also implies that we can assume

$$X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots \subset X$$

where $X^0 = X^1 = \cdots = X^{n-1}$ is a point. Since

$$\pi_n(X^{n+1}) = \pi_n(X), \quad \mathrm{H}_n(X^{n+1}) = \mathrm{H}_n(X),$$

we can further assume $X = X^{n+1}$.

By assumption

$$X^n = \bigvee_i S^n$$

is a wedge of spheres. Let

$$\varphi : \coprod_\alpha S^n \to \bigvee_i S^n$$

be the gluing map for attaching $(n+1)$-cells. Using the cellular approximation, we can assume φ is based

$$\varphi : Y = \bigvee_\alpha S^n \to \bigvee_i S^n.$$

Let

$$Z = M_\varphi$$

be the reduced mapping cylinder of φ, which is homotopy equivalent to X^n. X is the cofiber of φ

$$X = Z/Y.$$

We have the push-out diagram

$$\begin{array}{ccc} Y & \xrightarrow{p} & \star \\ {\scriptstyle j}\downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where $\star$ is the base point.

Replacing $\star$ by the reduced cone $C_\star Y$ and consider the push-out

$$\begin{array}{ccc} Y & \xrightarrow{\ i\ } & C_\star Y \\ {\scriptstyle j}\downarrow & & \downarrow \\ Z & \longrightarrow & \tilde{X} \end{array}$$

Since j is an $(n-1)$-equivalence and i is an n-equivalence, Homotopy Excision Theorem implies

$$\pi_n(Z, Y) = \pi_n(\tilde{X}, C_\star Y) = \pi_n(X) \quad \text{if} \quad n > 1.$$

This implies the exact sequence

$$\pi_n(Y) \to \pi_n(Z) \to \pi_n(X) \to \pi_{n-1}(Y) = 0 \quad \text{if} \quad n > 1.$$

For the case $n = 1$, Seifert-van Kampen Theorem implies that $\pi_1(X)$ is the quotient of $\pi_1(Z)$ by the normal subgroup generated by the image of $\pi_1(Y)$.

On the other side, we have the homology exact sequence

$$\mathrm{H}_n(Y) \to \mathrm{H}_n(Z) \to \mathrm{H}_n(Z, Y) = \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(Y) = 0.$$

Now we consider the commutative diagram

$$\begin{array}{ccccccc} \pi_n(Y) & \longrightarrow & \pi_n(Z) & \longrightarrow & \pi_n(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{H}_n(Y) & \longrightarrow & \mathrm{H}_n(Z) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & 0 \end{array}$$

Since Y, Z are wedges of spheres,

$$\pi_n(Y) \to \mathrm{H}_n(Y) \quad \text{and} \quad \pi_n(Z) \to \mathrm{H}_n(Z)$$

are abelianization homomorphisms. Therefore $\pi_n(X) \to \mathrm{H}_n(X)$ is also the abelianization homomorphism. $\square$

Example

The homology of S^n and Hurewicz Theorem implies that

$$\pi_k(S^n) = \begin{cases} 0 & \text{if} \quad k < n \\ \mathbb{Z} & \text{if} \quad k = n. \end{cases}$$

In particular, the degree of a map $f\colon S^n \to S^n$ can be described by either homotopy or homology.

Hurewicz Theorem has a relative version as well.

Theorem

Let (X, A) be a pair of path-connected spaces and A non-empty. Assume (X, A) is $(n-1)$-connected $(n \geq 2)$ and A is simply-connected. Then

$$\mathrm{H}_n(X, A) = 0 \quad \text{for} \quad i < n$$

and the Hurewicz map

$$\pi_n(X, A) \to \mathrm{H}_n(X, A)$$

is an isomorphism.

Theorem (Homology Whitehead Theorem)

Let $f: X \to Y$ between simply connected CW complexes. Assume

$$f_* : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

is an isomorphism for each n. Then f is a homotopy equivalence.

Proof

We can assume X is a CW subcomplex of Y. Then

$$\mathrm{H}_n(Y, X) = 0 \quad \text{for all} \quad n.$$

By Hurewicz Theorem,

$$\pi_n(Y, X) = 0 \quad \text{for all} \quad n.$$

Therefore f is weak homotopy equivalence, hence a homotopy equivalence by Whitehead Theorem.

□

Proposition

Every simply connected and orientable closed 3-manifold is homotopy equivalent to S^3.

Proof: Let X be a simply connected and orientable closed 3-manifold. Then

$$\mathrm{H}_0(X) = \mathbb{Z}, \quad \mathrm{H}_1(X) = \pi_1(X) = 0.$$

Since X is orientable, $\mathrm{H}_3(X) = \mathbb{Z}$ and Poincare duality holds (we will discuss in details later)

$$\mathrm{H}_2(X) = \mathrm{H}^1(X).$$

By the Universal Coefficient Theorem,

$$\mathrm{H}^1(X) = \mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_0(X), \mathbb{Z}) = 0.$$

So $\mathrm{H}_2(X) = 0$. By Hurewicz Theorem,

$$\pi_3(X) \to \mathrm{H}_3(X)$$

is an isomorphism. Let $f: S^3 \to X$ represent a generator of

$$\pi_3(X) = \mathbb{Z}.$$

Then

$$f_* : \mathrm{H}_\bullet(S^3) \to \mathrm{H}_\bullet(X)$$

are isomorphisms. It follows that f is a homotopy equivalence. □

Remark

The famous Poincare Conjecture asks that such X is homeomorphic to S^3.